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ON THE PECULIARITIES OF THE LOSS OF STABILITY OF A NON-LINEAR ELASTIC RECTANGULAR BAR†

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The stability of equilibrium of a compressed rectangular bar made of an isotropic incompressible elastic material that satisfies the Hadamard condition is investigated. It is found that the qualitative behaviour of the bar depends on whether the material belongs to one of three classes, conventionally referred to as materials of low, moderate, and high stiffness. For materials of low stiffness the equilibrium of an arbitrarily thick bar undergoes a bifurcation at a finite value of the critical deformation. For materials of moderate stiffness the critical deformation increases without limit as the relative thickness of the bar increases. For materials of high stiffness a "limiting" thickness exists, above which no bifurcation of the equilibrium is possible.

Simple criteria are obtained which enable one to determine the class to which the specific material belongs. It is established that the proposed classification of incompressible elastic materials is complete and consistent. Necessary and sufficient conditions for the existence of symmetric and antisymmetric modes of stability loss are found for materials of moderate and high stiffness. It is pointed out that, in some cases, symmetric bifurcation occurs prior to the antisymmetric one. For materials of low stiffness it is found that (for certain values of the relative thickness) double critical values of the deformation parameter may exist corresponding to two distinct buckling modes, namely, symmetric and antisymmetric ones. A sufficient condition for a symmetric bifurcation to be preceded by an antisymmetric one is stated. Specific models of incompressible elastic materials are considered. It is pointed out that the method developed in this paper is also effective in analysing the axisymmetric instability of a circular plate compressed by a uniform side pressure.

1. FORMULATION AND SOLUTION OF THE BOUNDARY-VALUE PROBLEM

CONSIDER the plane uniform deformation of an infinite bar of rectangular cross-section, the lateral sides $x=\pm a$ of the beam being acted upon by a uniformly distributed normal load of intensity q (per unit area of the surface of the configuration subject to deformation)

$$\begin{gathered} X=\lambda x, \quad Y=\lambda^{-1}y, \quad Z=z \quad (\lambda=\text{const}) \\ |x|\leqslant a, \quad |y|\leqslant h, \quad -\infty<z<+\infty \end{gathered} \tag{1.1}$$

It is assumed that no mass forces exist and the front sides $y=\pm h$ are stress-free. Moreover, the bar is assumed to be made of an isotropic incompressible material. In (1.1) x, y, z and X, Y, Z are the Cartesian coordinates before and after deformation, respectively. The parameters q and λ are related by the expression

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$$q = 2(c_1 + c_2)(\lambda^{-2} - \lambda^2)$$
$$c_m \equiv \partial\Pi/\partial I_m \quad (m = 1, 2) \tag{1.2}$$

Here Π is the volume density of the potential energy of deformation and I_m ($I = 1, 2$) are the first and second principal invariants of Finger's deformation measure [1] (for an incompressible material the third invariant I_3 is equal to unity). In what follows Π is assumed to satisfy the Hadamard condition [1, 2] and to be twice continuously differentiable as a function of I_1 and I_2 everywhere, except perhaps at the point $I_1 = I_2 = 3$, which corresponds to the non-deformed state (thus included into consideration are physically essentially non-linear materials for which the equation of state does not admit of linearization in the neighbourhood of the non-deformed state).

We will denote by D the region of admissible values of I_1 and I_2, which can be shown to be defined by

$$D = \{(I_1, I_2)\colon\ I_1 \geqslant 3,\ \ I_2 \geqslant 3,\ \ 4(I_1^2 - 3I_2)^3 - (9I_1I_2 - 2I_1^3 - 27)^2 \geqslant 0\}$$

for an incompressible material.

Let I be the rate in D corresponding to the uniform deformation (1.1) and defined parametrically by $I_1 = I_2 = \lambda^2 + \lambda^{-2} + 1$ $(0 < \lambda < 1)$. We shall assume that the inequality

$$c_1 + c_2 > 0 \quad ((I_1, I_2) \in I) \tag{1.3}$$

is satisfied for every point of I.

Condition (1.3) requires that the shear modulus of the material must be positive for any small simple shear deformation of the state of equilibrium in the XY plane described by (1.1), and is the same as one of the Baker–Ericksen inequalities [1, 2] (for points belonging to I). Note that (1.3) does not contradict the Hadamard condition, since the latter implies that $c_1 + c_2 \geqslant 0$ $((I_1, I_2) \in D)$, which is a weaker version of (1.3).

Within the framework of the static approach we shall study the stability of the equilibrium configuration (1.1) with respect to small plane perturbations. In the case of a plane deformation, the equilibrium equations, linearized in the neighbourhood of the state (1.1), have the form $(\gamma \equiv \lambda^{-2})$

$$\begin{aligned} &[(1+\varepsilon)\partial_1^2 + \partial_2^2]u + \lambda^{-1}\partial_1 p = 0 \\ &(\partial_1^2 + \partial_2^2)\upsilon + \lambda\partial_2 p = 0 \\ &\gamma\partial_1 u + \partial_2\upsilon = 0 \end{aligned} \tag{1.4}$$

$$\varepsilon = \frac{2\lambda^2(1-\gamma^2)^2(c_{11} + 2c_{12} + c_{22})}{c_1 + c_2}, \quad c_{km} \equiv \frac{\partial^2\Pi}{\partial I_k \partial I_m} \quad (k, m = 1, 2) \tag{1.5}$$

Here u and υ are the projections of the displacement vector onto the X and Y axes of the Cartesian system of coordinates and ∂_i $(i = 1, 2)$ are the operators of differentiation with respect to x and y, respectively. Since the material is incompressible, the linearized equilibrium equations (1.4) contain an unknown function p of the coordinates, which has the dimensions of pressure and can be determined in the course of solving the problem. The latter equation in (1.4) is the linearized incompressibility condition.

Equations (1.4) can be derived by means of the theory of superimposing a small deformation on a finite one [1].

The components of the linearized Piola stress tensor [1, 2] can be expressed in terms of the displacements u and υ and the pressure p

$$
\begin{aligned}
&g^{-1}P_{11} = (1+\gamma^2+\varepsilon)\partial_1 u + \lambda^{-1}p, \qquad g^{-1}P_{12} = \partial_1 \upsilon + \gamma\partial_2 u \\
&g^{-1}P_{21} = \gamma\partial_1\upsilon + \partial_2 u, \qquad g^{-1}P_{22} = 2\partial_2\upsilon + \lambda p \\
&g^{-1}P_{33} = (\delta + 2c)\partial_1 u + p, \qquad P_{i3} = P_{3i} = 0 \qquad (i = 1, 2)
\end{aligned} \tag{1.6}
$$

$$
\begin{aligned}
&\delta = 4g^{-1}(\lambda + \lambda^{-1})(1-\gamma^2)^2[c_{11} + c_{12}(1+\lambda^2) + c_{22}\lambda^2] \\
&g = 2(c_1 + c_2), \qquad c = 2g^{-1}c_2\lambda(1-\gamma^2)
\end{aligned} \tag{1.7}
$$

According to (1.6), the linearized boundary conditions on the front sides of the bar can be expressed in the form

$$
(\gamma\partial_1\upsilon + \partial_2 u)\,|_{y=\pm h} = 0, \qquad (2\partial_2\upsilon + \lambda p)\,|_{y=\pm h} = 0 \tag{1.8}
$$

where h is the half-thickness of the bar before deformation. The "sliding" clamping conditions are satisfied on the lateral sides $x=\pm a$ [3, 4], i.e. no normal displacement or shear stresses exist

$$
u\,|_{x=\pm a} = 0, \qquad (\partial_1\upsilon + \gamma\partial_2 u)\,|_{x=\pm a} = 0 \tag{1.9}
$$

It is obvious that the boundary conditions (1.9) can be satisfied by seeking u, υ, and p in one of the following forms

$$
u = U(y)\sin kx, \quad \upsilon = V(y)\cos kx, \quad p = \lambda P(y)\cos kx \tag{1.10}
$$

$$
u = U(y)\cos lx, \quad \upsilon = V(y)\sin lx, \quad p = \lambda P(y)\sin lx \tag{1.11}
$$

The parameters k and l can be determined from the conditions $\sin ka = 0$ and $\cos la = 0$, i.e.

$$
k = k_m = \frac{\pi m}{a}, \quad l = l_m = \frac{\pi(2m-1)}{2a} \quad (m = 1, 2, 3, \ldots) \tag{1.12}
$$

Substituting (1.10) into (1.4) and (1.8), we obtain the homogeneous boundary-value problem

$$
\begin{aligned}
&U'' - (1+\varepsilon)k^2U - kP = 0 \\
&V'' - k^2V + \lambda^2P' = 0 \\
&\gamma kU + V' = 0
\end{aligned} \tag{1.13}
$$

$$
(U' - \gamma kV)\,|_{y=\pm h} = 0, \qquad (2V' + \lambda^2P)\,|_{y=\pm h} = 0 \tag{1.14}
$$

for U, V, and P, the prime denoting differentiation with respect to y. Obviously, problem (1.13), (1.14) admits of two types of solutions. For one of them the deflection amplitude $V(y)$ is an even function of y, while being an odd function for the other one. Moreover, the loss of stability has, respectively, an antisymmetric or symmetric form relative to the middle plane $y=0$ of the bar. Consequently, it is natural to say that a solution of the first type is antisymmetric and a solution of the other type is symmetric. An antisymmetric solution corresponds to the bending forms of loss of stability of the bar.

Leaving out the intermediate discussion, we will state the final form of the solution of the boundary-value problem (1.13), (1.14) for the two cases at hand

$$
\begin{aligned}
&U = \beta[\omega_2(1+\omega_1^2)\Phi_1^{\pm}(y) - \omega_1(1+\omega_2^2)\Phi_2^{\pm}(y)] \\
&V = \beta[(\omega_2^2+\gamma^2)\Psi_1^{\mp}(y) - (\omega_1^2+\gamma^2)\Psi_2^{\mp}(y)] \\
&P = k\beta[\omega_2^3(\omega_1^4-1)\Phi_1^{\pm}(y) - \omega_1^3(\omega_2^4-1)\Phi_2^{\pm}(y)]
\end{aligned} \tag{1.15}
$$

$$\beta[(\omega_1^2+\gamma^2)^2\omega_2\Phi_1^{\mp}(h)-(\omega_2^2+\gamma^2)^2\omega_1\Phi_2^{\mp}(h)]=0 \tag{1.16}$$

$$\Phi_m^+(y)=\frac{\operatorname{sh}(\omega_m ky)}{\operatorname{ch}(\omega_m kh)},\quad \Phi_m^-(y)=\frac{\operatorname{ch}(\omega_m ky)}{\operatorname{sh}(\omega_m kh)}$$

$$\Psi_m^+(y)=\frac{\operatorname{sh}(\omega_m ky)}{\operatorname{sh}(\omega_m kh)},\quad \Psi_m^-(y)=\frac{\operatorname{ch}(\omega_m ky)}{\operatorname{ch}(\omega_m kh)}$$

$$(m=1,2)$$

The minus and plus superscripts in (1.15) and (1.16) correspond to the antisymmetric solution, while the subscripts correspond to the symmetric one. Here

$$\omega_{1,2}=\tfrac{1}{2}(\sqrt{\mu+2\gamma}\pm\sqrt{\mu-2\gamma})$$
$$\mu=1+\gamma^2+\varepsilon,\quad \beta=(\omega_1^2-\omega_2^2)^{-1} \tag{1.17}$$

Formulae (1.15) and (1.16) are suitable for $\omega_1\neq\omega_2$. The case of equal characteristic numbers ($\omega_1=\omega_2\equiv\omega$) can be obtained from (1.15) and (1.16) by taking the limit as $\omega_1\to\omega$ and $\omega_2\to\omega$.

If the unknowns u, υ, and p are sought in the form (1.11), the corresponding boundary-value problem for the amplitude functions U, V, and P and its solutions in the antisymmetric and symmetric cases can be obtained from (1.13)–(1.16) by the substitution $k\sim -l$. For brevity, the relations obtained in this way will be said to be concomitant with (1.13)–(1.16).

The two equations (1.16) (as well as the concomitant equations) define the bifurcation values of γ for the antisymmetric and symmetric forms of loss of stability, respectively, and will henceforth be called the characteristic equations.

2. ANALYSIS OF THE CHARACTERISTIC EQUATIONS. CLASSIFICATION OF ISOTROPIC INCOMPRESSIBLE ELASTIC MATERIALS

On substituting $k=k_m$ and $l=l_m$, respectively, into (1.16) and into the equation concomitant with (1.16) (in accordance with (1.12)), one can express the characteristic equations as follows:

$$\beta[(\omega_1^2+\gamma^2)^2\omega_2\operatorname{cth}\omega_1\sigma_m\tau-(\omega_2^2+\gamma^2)^2\omega_1\operatorname{cth}\omega_2\sigma_m\tau]=0 \tag{2.1}$$

$$\beta[(\omega_1^2+\gamma^2)^2\omega_2\operatorname{th}\omega_1\sigma_m\tau-(\omega_2^2+\gamma^2)^2\omega_1\operatorname{th}\omega_2\sigma_m\tau]=0 \tag{2.2}$$

$$\tau\equiv h/a,\quad \sigma_m=\pi m/2\qquad (m=1,2,3,\ldots) \tag{2.3}$$

The first of these equations corresponds to the antisymmetric case. The second equation corresponds to the symmetric case. If m is even, the critical values of γ defined by (2.1) and (2.2) correspond to buckling modes of the type (1.10) (where $k=\sigma_m/a$). If m is odd, the critical values of γ correspond to buckling modes of the type (1.11) (where $l=\sigma_m/a$).

An analysis of (2.1) and (2.2) reveals that, subject to the restrictions on the potential Π adopted in Sec. 1, the following three essentially different alternatives are possible:

1. a bifurcation of the equilibrium state (1.1) of the bar occurs for any value of the relative thickness τ, the loss of stability of an arbitrarily thick bar occurring at a finite value of the critical deformation γ_*;

2. a bifurcation of the equilibrium of the bar also occurs for any τ, but the critical deformation γ_* increases without limit as τ increases to ∞;

3. it is impossible for the equilibrium of the bar to undergo a bifurcation for $\tau>\tau_*$, where τ_* is a fixed value of τ (called henceforth the "limiting" thickness); if $\tau\leqslant\tau_*$, then, as a rule, a

bifurcation of the equilibrium occurs (although exceptions are possible).

The first situation is typical of the majority of known models of incompressible elastic materials [1, 5, 6] (the Treloar, Mooney–Rivlin, Bartenev–Khazanovich, Klosner–Segal, Hutchinson–Becker–Lendel, and other models). Examples of potentials for which the second and third cases occur are presented in Sec. 3.

As a consequence, it is found to be useful to divide all isotropic incompressible elastic materials into three groups. For brevity, we shall refer to them as materials of low, moderate, and high stiffness (in accordance with cases 1–3).

To give a precise formulation of the results obtained, we will introduce the following notation

$$R(\gamma) = \gamma^3 - 2\gamma^2 - \gamma - \mu(\gamma), \quad \gamma \in L \equiv (1, +\infty) \tag{2.4}$$

$$\Gamma_R^+ = \{\gamma \in L:\ R(\gamma) > 0\}, \quad \Gamma_R^- = \{\gamma \in L:\ R(\gamma) < 0\}$$
$$\Gamma_R^0 = \{\gamma \in L:\ R(\gamma) = 0\} \tag{2.5}$$

$$\Delta(\gamma) = \mu(\gamma) - 2\gamma, \quad \Sigma(\gamma) = \mu(\gamma) + 2\gamma, \ \ \gamma \in L \tag{2.6}$$

$$\Gamma_\Delta^- = \{\gamma \in L:\ \Delta(\gamma) < 0\} \tag{2.7}$$

$$F(\gamma) = \sqrt{\frac{|\Delta|}{\Sigma}}\operatorname{arch} q_1 - \arccos q_2, \quad \gamma \in \overline{\Gamma_\Delta^-} \setminus \Gamma_R^0 \tag{2.8}$$

$$q_1 = \frac{\gamma(\gamma^2 + 1 + \mu)}{|\gamma(\gamma - 1)^2 - \Sigma|}, \quad \gamma \in L \setminus \Gamma_R^0; \quad q_2 = \frac{\gamma(\gamma^2 + 1 + \mu)}{|\gamma(\gamma + 1)^2 + \Delta|}, \quad \gamma \in L$$
$$G(\gamma) = F(\gamma) - \pi, \quad \gamma \in \overline{\Gamma_\Delta^-} \setminus \Gamma_R^0 \tag{2.9}$$

$$\Gamma_G^+ = \{\gamma \in \overline{\Gamma_\Delta^-} \setminus \Gamma_R^0:\ G(\gamma) \geqslant 0\} \tag{2.10}$$

A bar denotes the closure of a set and a slash denotes the set-theoretic difference.

Theorem 1. We will assume that the potential Π satisfies the Hadamard condition (1.3) and the restrictions

$$R(\gamma) < 0, \quad \gamma \in L \tag{2.11}$$

$$\varliminf_{\gamma \to 1+0} \frac{\Sigma(\gamma)}{\gamma - 1} > 0 \tag{2.12}$$

$$\varliminf_{\gamma \to +\infty} \frac{\mu(\gamma)}{\gamma^4} > 0 \tag{2.13}$$

Then a "limiting" thickness τ_* exists above which it is impossible for the equilibrium of the homogeneous configuration (1.1) to undergo a bifurcation, i.e. neither of the characteristic equations (2.1), (2.2) has a solution with respect to γ (for all $m \geqslant 1$). But if $\tau \leqslant \tau_*$, the following assertions hold.

1. For a given $m \geqslant 1$, the characteristic equation (2.1) is solvable with respect to γ if and only if

$$m\tau \in T^- \setminus \{0\} \tag{2.14}$$

where T^- is a closed bounded set consisting of a finite or denumerable family of disjoint intervals

$$T^- = \bigcup_{n=1}^{n^-} [t_n^-, \theta_n^-] \quad (1 \leqslant n^- \leqslant +\infty) \tag{2.15}$$

$$t_1^- = \inf T^- \geqslant 0, \quad \sup T^- \leqslant \tau_*$$
$$t_{n-1}^- \leqslant \theta_{n-1}^- < t_n^- \leqslant \theta_n^- \quad (n = 2,3,\ldots,n^-)$$

The set T^- is always non-empty, but does not have to cover the interval $[0, \tau_*]$. The equality $t_1^- = 0$ holds, subject to the condition

$$\left[\overline{\lim_{\gamma\to 1+0}} \frac{\Sigma(\gamma)}{\gamma - 1} = +\infty\right] \vee \left[\overline{\lim_{\gamma\to+\infty}} \frac{\mu(\gamma)}{\gamma^4} = +\infty\right] \tag{2.16}$$

Otherwise $t_1^- > 0$.

2. If

$$(\Gamma_\Delta^- = \phi) \vee (\Gamma_G^+ = \phi) \tag{2.17}$$

(see the notation (2.7)–(2.10)), then the characteristic equation (2.2) has no solutions with respect to γ for any $m \geqslant 1$. Moreover, T^- consists of a single interval

$$T^- = [t_1^-, \theta_1^-], \quad t_1^- \geqslant 0, \quad \theta_1^- = \tau_*$$

3. If the requirement

$$\Gamma_G^+ \neq \phi \tag{2.18}$$

is satisfied, then a closed bounded set T^+ exists that can be represented as a finite or denumerable union of disjoint intervals

$$T^+ = \bigcup_{n=1}^{n^+} [t_n^+, \theta_n^+] \quad (1 \leqslant n^+ \leqslant +\infty) \tag{2.19}$$
$$t_1^+ = \inf T^+ > 0, \quad \sup T^+ \leqslant \tau_*$$
$$t_{n-1}^+ \leqslant \theta_{n-1}^+ < t_n^+ \leqslant \theta_n^+ \quad (n = 2,3,\ldots,n^+)$$

such that, for any given $m \geqslant 1$, the characteristic equation (2.2) is solvable if and only if

$$m\tau \in T^+ \tag{2.20}$$

4. For a given $\tau > 0$, antisymmetric buckling modes exist if and only if there is a number $m \geqslant 1$ such that the condition (2.14) is satisfied. In particular, under the conditions (2.16) and (2.17), antisymmetric modes exist for all $\tau \leqslant \tau_*$ (and only for such τ).

5. For a given $\tau > 0$, symmetric buckling modes exist if and only if condition (2.18) is satisfied and there is a number $m \geqslant 1$ such that (2.20) is satisfied. In particular, if condition (2.17) is satisfied, no symmetric forms exist for any $\tau > 0$.

Remarks on Theorem 1. 1. If requirement (2.18) is satisfied, the symmetric buckling modes may occur prior to the antisymmetric ones. We also observe that the inequality $\Delta(\gamma) \geqslant 0$ $(\gamma \in L)$ is a sufficient condition for the non-existence of symmetric modes.

2. In the general case, the equality $T^+ \cup T^- = [0, \tau_*]$ is not true, i.e. for $\tau \leqslant \tau_*$, a τ may exist such that Eqs (2.1) and (2.2) have no solutions and a bifurcation of equilibrium is impossible.

3. In order to make the formulation of the theorem more compact in the exceptional cases when $m\tau = \tau_*$ and $\mu(\gamma)/\gamma^4$ has a finite limit as $\gamma \to +\infty$, the "limiting" solution $\gamma = +\infty$, which is sometimes possible, is also included among the solutions of Eq. (2.1). The latter solution has no mechanical meaning and can be easily discarded in any specific situation.

4. Situations when the "limiting" thickness τ_* is isolated in the following sense are included: a

neighbourhood W_* of τ_* exists such that, for $\tau \in W_*$, a bifurcation of the equilibrium of the bar occurs if and only if $\tau = \tau_*$.

5. The critical values of γ corresponding to the symmetric modes of stability loss (provided the latter exist) lie inside the interval (1, 3).

6. It can be shown that the sets $T^{\pm}$ consist of a finite system of intervals if the equation $F(\gamma) = n\pi$ has not more than a finite number of solutions on any bounded subset of L for any $n \geqslant 1$. For brevity, we shall refer to the latter restriction on the potential Π as the F-condition. The geometric meaning of the F-condition is that the set of branch points of the real spectral curves $k = k(\gamma)$ of the boundary-value problem (1.13), (1.14) has no accumulation points in any bounded region in the half-plane $\gamma > 1$, $|k| < +\infty$. Physically, it can be regarded as the requirement that the mechanical characteristics of the material should vary more or less smoothly during the deformation. As a rule, the F-condition is satisfied. It may be violated only in individual cases of no practical interest (for example, for oscillatory potentials Π, which can hardly pretend to provide an adequate description of the behaviour of real materials). Thus the sets $T^{\pm}$ are most frequently formed by a finite number of intervals.

The proof of Theorem 1 (and all subsequent results) is very long and will therefore be omitted. We will merely observe that the critical values of γ can obviously be determined in two stages. Firstly, one can construct the functions $\theta = \theta(\gamma)$ starting from the equations

$$\beta[(\omega_1^2 + \gamma^2)^2 \omega_2 \operatorname{cth} \omega_1 \theta - (\omega_2^2 + \gamma^2)^2 \omega_1 \operatorname{cth} \omega_2 \theta] = 0 \tag{2.21}$$

$$\beta[(\omega_1^2 + \gamma^2)^2 \omega_2 \operatorname{th} \omega_1 \theta - (\omega_2^2 + \gamma^2)^2 \omega_1 \operatorname{th} \omega_2 \theta] = 0 \tag{2.22}$$

Then one can find the points of intersection of the curves (in the (γ, θ) plane) with the straight lines

$$\theta = \sigma_m \tau \qquad (m = 1, 2, 3, \ldots) \tag{2.23}$$

The numbers σ_m $(m = 1, 2, 3, \ldots)$ are determined by (2.3). The x-coordinates of the points in question are equal to the bifurcation values of γ. The problem can therefore be reduced to a study of the position of the spectral curves of Eqs (2.21) and (2.22) Since the right-hand sides in (2.23) are real valued, it is possible to confine oneself to the study of the real part of the spectrum only. For special models of incompressible materials, a similar analysis of Eqs (2.21) and (2.22) (which are exactly the same as the characteristic equations of the theory of uniform solutions for prestressed elastic plates) can be found, for example, in [7, 8] (the Treloar and Mooney–Rivlin materials), and [9] (the Bartenev–Khazanovich and Chernykh–Shubina materials).†

Theorem 2. Let all the requirements of the premise of Theorem 1 be satisfied, except for inequality (2.13), instead of which the condition

$$\varliminf_{\gamma \to +\infty} \frac{\mu(\gamma)}{\gamma^4} = 0 \tag{2.24}$$

is satisfied. Then a bifurcation of the equilibrium of a compressed bar occurs for all $\tau > 0$, the critical value $\gamma_*(\tau)$ of γ increasing without limit as the thickness τ increases to infinity. Moreover, the following assertions hold:

1. when the condition

$$\varlimsup_{\gamma \to 1+0} \frac{\Sigma(\gamma)}{\gamma - 1} = +\infty \tag{2.25}$$

†See also: RUDEV A. N., Homogeneous solutions for an elastic plate in the case of an affine initial deformation. Rostov-on-Don, 1980. Unpublished paper, deposited at VINITI 04.07.80, No. 3937-80.

is satisfied the characteristic equation (2.1) is solvable with respect to γ for any $m \geqslant 1$ and $\tau > 0$;

2. if condition (2.25) is violated for any fixed $\tau > 0$ then an integral number $m_*(\tau) \geqslant 1$ exists such that the characteristic equation (2.1) has no solutions for $m < m_*(\tau)$ and has at least one solution for $m \geqslant m_*(\tau)$;

3. if the requirement (2.17) is met, the characteristic equation (2.2) has no solutions for any $m \geqslant 1$ and $\tau > 0$;

4. assertion 3 in the conclusion of Theorem 1 holds;

5. antisymmetric buckling modes exist for any $\tau > 0$;

6. assertion 5 in the conclusion of Theorem 1 holds.

Remarks on Theorem 2. 1. As a rule, for any τ, the bifurcation value γ_* of γ fails to be uniquely defined. Since the assertion of the theorem concerning the behaviour of $\gamma_*(\tau)$ as $t \to \infty$ is independent of the choice of γ_*, it follows that it is correct. In particular, $\gamma_*(\tau)$ can be understood to be the least critical value of γ for the thickness τ in question. In this case $\gamma_*(\tau)$ is, in general, a piecewise-continuous function of τ on the ray $(0, +\infty)$.

2. Remarks 5 and 6 on Theorem 1 remain valid under the assumptions of Theorem 2 (the latter remark applies to T^+).

Theorem 3. Assume that the following conditions are satisfied:

1. the potential Π satisfies the Hadamard condition and the restriction (1.3);

2. the function $R(\gamma)$ has an isolated zero γ_0 on L; moreover, the set Γ_R^0 is finite, and the inequalities $(\gamma_0 = \sup \Gamma_R^0,\ \gamma_0 < \gamma_1 \leqslant +\infty,\ 0 < \alpha < \gamma^0 - 1)$

$$\begin{aligned} &R(\gamma) < 0, \quad \gamma \in L^- \equiv (1, \gamma_0); \quad R(\gamma) > 0, \quad \gamma \in L^+ \equiv (\gamma_0, \gamma_1) \\ &R(\gamma) < 0, \quad \gamma \in (\gamma^0 - \alpha, \gamma^0); \quad R(\gamma) > 0, \quad \gamma \in (\gamma^0, +\infty) \end{aligned} \tag{2.26}$$

are satisfied;

3. if $\Delta(\gamma_0) = 0$, then γ_0 fails to be a limiting point of the set Γ_G^+;

4. $\Sigma(\gamma)$ satisfies (2.12);

5. if $\Delta(\gamma_0) < 0$, then: (a) the inequality $\inf \Gamma_G^+ > 1$; is satisfied; (b) a neighbourhood W of γ_0 exists such that $R(\gamma)$ is a continuously differentiable function in W and $\dot{R}(\gamma_0) > 0$ (the dot denotes the derivative with respect to γ).

Then a bifurcation of the equilibrium state (1.1) of the bar occurs for all $\tau > 0$. Moreover, the least critical value γ_* of γ does not exceed γ_0

$$\gamma_* \leqslant \gamma_0$$

In addition, the following assertions are true.

1. Antisymmetric and symmetric buckling modes exist for any thickness $\tau > 0$.

2. Under condition (2.25), the characteristic equation (2.1) is solvable (with respect to γ) for any $m \geqslant 1$ and $\tau > 0$. If the condition (2.25) is violated, a sequence of "minimum" thicknesses τ_m^- $(m = 1, 2, 3, \ldots)$ exists such that, for a fixed $m \geqslant 1$, Eq. (2.1) is solvable if and only if $\tau \geqslant \tau_m^-$. The sequence τ_m^- $(m = 1, 2, 3, \ldots)$ decreases monotonically and converges to zero

$$\lim_{m \to \infty} \tau_m^- = 0 \tag{2.27}$$

3. If the requirement

$$\lim_{\gamma \to +\infty} \frac{|\mu(\gamma)|}{\gamma^{8/3}} = 0 \tag{2.28}$$

is met, the characteristic equation (2.2) is solvable for any $m \geqslant 1$ and $\tau > 0$. But if (2.28) is violated, a sequence τ_m^+ ($m=1, 2, 3, \ldots$) of "minimum" thicknesses exists such that, for a fixed $m \geqslant 1$, Eq. (2.2) is solvable if and only if $\tau \geqslant \tau_m^+$. The sequence τ_m^+ ($m=1, 2, 3, \ldots$) decreases monotonically and converges to zero

$$\lim_{m\to\infty} \tau_m^+ = 0 \tag{2.29}$$

4. If the inequality

$$\Delta(\gamma_0) \geqslant 0 \tag{2.30}$$

is satisfied, then, for any $m \geqslant 1$, a value τ_m of τ exists such that, for $\tau \geqslant \tau_m$, the least solutions $\gamma_m^-(\tau)$ and $\gamma_m^+(\tau)$ of Eqs (2.1) and (2.2), respectively, satisfy the relation

$$\gamma_m^\pm(\tau) \in L^\pm \tag{2.31}$$

The sequence τ_m ($m=1, 2, 3, \ldots$) decreases monotonically and converges to zero. But if the condition

$$\Delta(\gamma_0) < 0 \tag{2.32}$$

is satisfied, an integral number $N \geqslant 0$ exists such that the following relations hold

$$\gamma_m^\pm(\tau_m^{(n)}) = \gamma_0 \quad (m \geqslant 1,\ n > 2N) \tag{2.33}$$

$$\tau \in (\tau_m^{(2r)}, \tau_m^{(2r+1)}) \Rightarrow \gamma_m^\pm(\tau) \in L^\pm \quad (m \geqslant 1,\ r \geqslant N) \tag{2.34}$$

$$\tau \in (\tau_m^{(2r+1)}, \tau_m^{(2r+2)}) \Rightarrow \gamma_m^\pm(\tau) \in L^\mp \quad (m \geqslant 1,\ r \geqslant N) \tag{2.35}$$

$$\tau_m^{(n)} \equiv \frac{2n}{m\sqrt{|\Delta(\gamma_0)|}} \quad (m \geqslant 1,\ n \geqslant 0) \tag{2.36}$$

In (2.31), (2.34), and (2.35) the plus and minus indices appear simultaneously either as superscripts or subscripts.

5. If the "minimum" thicknesses τ_m^+ and τ_m^- exist and the inequality (2.32) is satisfied, then

$$\tau_m^\pm \leqslant \frac{2}{m\sqrt{|\Delta(\gamma_0)|}} \quad (m \geqslant 1) \tag{2.37}$$

6. If the function $\Delta(\gamma)$ is non-negative for $\gamma > 1$ and $\gamma_1 = +\infty$, then (2.31) holds for all $m \geqslant 1$ and $\tau > 0$, i.e. the antisymmetric modes must precede the symmetric ones.

Remarks on Theorem 3. 1. The mechanical meaning of the "minimum" thicknesses is that, for some materials, symmetric (or antisymmetric) buckling modes with m nodal lines exist only for $\tau \geqslant \tau_m^+$ (or $\tau \geqslant \tau_m^-$). If, for a bar made of such a material, the relative thickness τ is fixed, then, in view of (2.27) (or (2.28)), a symmetric (or antisymmetric) bifurcation always occurs, but the number of nodal lines cannot be arbitrary and must be greater than the least number m that satisfies the inequality $\tau \geqslant \tau_m^+$ (or $\tau \geqslant \tau_m^-$). As an example, one can consider the potential (3.15) under the condition $n \in (2/3, 1)$, in which case the "minimum" thicknesses τ_m^+ exist (see Sec. 3).

2. The inequality $\Delta(\gamma) \geqslant 0 (\gamma \in L)$ is a sufficient condition for the symmetric modes to be preceded by the antisymmetric ones, not only within the framework of Theorem 3, but also in the most general case when Π satisfies both the Hadamard condition and (1.3).

3. The conclusion of Theorem 3 remains valid if assumption (a) in Condition 5 is omitted and Condition 4 is replaced by

$$\lim_{\gamma \to 1+0} \frac{\Sigma(\gamma)}{\gamma - 1} \geqslant \sigma_* \tag{2.38}$$

Here $\sigma_* = 1.11008 \ldots$ is the unique positive solution of the equation $H(\sigma) = \pi$, where the function $H(\sigma)$ is defined by

$$H(\sigma) = \frac{2\sqrt{2(\sigma+2)}}{\sigma} - \arccos\frac{\sigma}{\sigma+4}, \quad \sigma \in (0, +\infty)$$

In some cases it may be preferable to verify (2.38).

Theorem 3 implies that if the condition (2.32) is satisfied and $\tau \in (\tau_m^{(2r+1)}, \tau_m^{2r+2})$ (at least for sufficiently large r), then the mth symmetric mode of stability loss can be observed at a lower critical deformation value than the mth antisymmetric mode. But since the least bifurcation value of γ is given by $\min_{m \geqslant 1}\{\gamma_m^- (\tau), \gamma_m^-(\tau)\}$ in the general case, the values of the relative thicknesses τ for which the minimum is attained for a symmetric buckling mode are not excluded. Consequently, Remark 1 on Theorem 1 remains valid under the assumptions of Theorem 3 (as well as Theorem 2). Moreover, a buckling mode with several nodal lines may correspond to the least critical value of γ.

We also remark that if the inequality (3.32) is satisfied, then γ_0 is a common solution of Eqs (2.1) and (2.2) for $\tau = \tau_m^{(n)}$ $(m, n \geqslant 1)$, i.e. γ_0 is a double eigenvalue of the boundary-value problem (1.4), (1.8), (1.9), to which there correspond two distinct buckling modes, namely, an antisymmetric and a symmetric one. On the basis of what has been said above, one can expect interesting effects in the analysis of the post-critical behaviour of an elastic body. However, this is a subject for a separate study.

Theorems 1–3 can obviously serve as sufficient criteria for a material to be of high, moderate, or low stiffness. It can be shown that the first two theorems give not only sufficient, but also necessary conditions. In view of this, a comparative analysis of Theorems 1–3 enables us to conclude that:

1. for materials of high stiffness, the possible buckling modes (both antisymmetric and symmetric ones) have a limited number of nodal lines for any fixed $\tau \leqslant \tau_*$;
2. for materials of moderate stiffness, the number of nodal lines can be as large as desired in the case of the antisymmetric modes, but it is bounded for the symmetric ones (for a fixed $\tau > 0$);
3. for materials of low stiffness (in the framework of Theorem 3), the existing buckling modes—both antisymmetric and symmetric ones—can have as large a number of nodal lines as desired for any $\tau > 0$.

Theorems 1–3, which reveal the most typical aspects of the behaviour of incompressible elastic materials (under the restrictions on Π adopted in Sec. 1), do not exhaust the whole variety of situations allowed by the Hadamard condition. Thus, the determinant function $R(\gamma)$ may have a zero at γ_0 without changing the sign (see example (2.44) below), or several zeros $\gamma_0^{(1)}, \gamma_0^{(2)}, \ldots, \gamma_0^{(n)}$ (either simple or of arbitrary multiplicity), or a denumerable set of zeros, or even a continuum of zeros (either bounded or unbounded). It is impossible to give a complete description of all admissible alternatives. Each case that fails to fit into the framework of Theorems 1–3 requires a separate discussion. Nevertheless, the following general result holds.

Theorem 4. Suppose that assumption 1 of Theorem 3 holds and one of the mutually contradictory conditions stated below is satisfied (see the notation (2.5))

$$\Gamma_R^0 \neq \phi \tag{2.39}$$

$$(\Gamma_R^- = L) \Rightarrow \left[\lim_{\gamma \to 1+0} \frac{\Sigma(\gamma)}{\gamma - 1} = 0\right] \tag{2.40}$$

$$\Gamma_R^+ = L \tag{2.41}$$

Then the following assertions hold:

1. for any $\tau > 0$, at least one of Eqs (2.1), (2.2) is solvable for γ, i.e. the equilibrium of the

homogeneous configuration (1.1) undergoes a bifurcation;

2. if $\gamma_*(\tau)$ is the least critical value of γ for the given thickness τ, then $\gamma_*(\tau)$ is a bounded function on the ray $(0, +\infty)$;

3. If condition (2.39) is satisfied, then $\gamma_*(\tau)$ satisfies the estimates

$$\gamma_*(\tau) \leqslant \inf \Gamma_R^0 \quad (\inf \Gamma_R^0 > 1)$$
$$\gamma_*(\tau) < 3 \quad (\inf \Gamma_R^0 = 1) \tag{2.42}$$

4. the relation

$$\gamma_*(\tau) \equiv 1, \qquad \tau \in (0, +\infty) \tag{2.43}$$

is satisfied if (3.29) is violated;

5. an unbounded set T^- of continuum intensity exists such that antisymmetric modes of stability loss exist for all $\tau \in T^-$;

6. if requirements (2.40) and (2.41) are satisfied, then there is an unbounded set T^+ of continuum intensity such that symmetric modes of stability loss exist for all $\tau \in T^+$;

7. if the sets T^+ and T^- exist simultaneously, then they can be chosen in such a way that $T^+ \cup T^- = (0, +\infty)$; otherwise one can assume that $T^- = (0, +\infty)$.

Remarks on Theorem 4. 1. In general, for a given $\tau > 0$, the set of critical values of γ does not always contain a minimum element. In such a case $\gamma_*(\tau)$ (which appears, in particular, in (2.42) and (2.43)) is to be understood as the infimum of this set.

2. If (2.39) is satisfied, then it is possible that no symmetric modes of stability loss exist for any $\tau > 0$. As an example, we mention the potential

$$\Pi(I_1, I_2) = d_1\Phi(J_1) + d_2\Phi(J_2) \tag{2.44}$$
$$(d_1 \geqslant 0, \ d_2 \geqslant 0, \ d_1^2 + d_2^2 \neq 0)$$
$$J_k = \tfrac{1}{2}(I_k - 1 + \sqrt{(I_k+1)(I_k-3)}) \quad (k = 1, 2)$$

where the function $\Phi(t)$ is given by the integral representation

$$\Phi(t) = \int_1^t \frac{(\theta-1)^{12}(\theta+1)^{27}}{\theta^{39}} e^{\theta/2} d\theta \quad (t \geqslant 1)$$

It can be verified that the potential (2.44) satisfies the Hadamard condition, restriction (1.3), and relation (2.39) with $\Gamma_R^0 = \{5\}$. Thus the characteristic equation (2.2) has no solutions for any $\tau > 0$ and $m \geqslant 1$.

3. We note that in the most general case (when Π satisfies the Hadamard condition and the restriction (1.3)) the necessary and sufficient conditions for the non-existence of symmetric buckling modes have the form

$$\Gamma_R^+ = \phi, \quad \Gamma_R^- \supset (1, 3), \quad \Gamma_R^- \cap \Gamma_G^+ = \phi \tag{2.45}$$

In particular, for the potential (2.44), we obtain $L_R^+ = \phi$, $\Gamma_R^- = L \setminus \{5\}$, and $\Gamma_G^+ = \phi$, i.e. the requirements (2.45) are satisfied.

4. It is obvious that Theorem 4 provides sufficient conditions for the material to be of low stiffness. It turns out that conditions (2.39)–(2.41) are not only sufficient, but also necessary. On taking account of what has been said above regarding Theorems 1 and 2 and comparing requirements (2.11)–(2.13), (2.24), and (2.39)–(2.41), we conclude that the above classification of incompressible elastic materials is complete and consistent.

We remark that the results obtained in Sec. 2 also hold in the case when Π satisfies the strong ellipticity condition [1, 2], since the latter ensures that the Hadamard condition and requirement (1.3) are both satisfied.

3. EXAMPLES

We shall consider a number of specific examples which illustrate the discussion in Sec. 2.

3.1. *A Hart–Smith material* [5, 10]

$$\Pi = d_1 \int e^{\nu(I_1-3)^2} dI_1 + d_2 \ln\frac{I_2}{3} \quad (d_1>0,\ d_2 \geqslant 0,\ \nu>0) \tag{3.1}$$

The potential (3.1) was obtained in [10] using the non-Gaussian molecular theory. One can prove the following sufficient criterion for the Hadamard condition to be satisfied.

Lemma 1. If the elastic constants d_1, d_2, and ν satisfy the inequalities

$$\frac{d_2}{d_1}<1, \quad \Phi(\nu)-\frac{d_2}{3d_1} \geqslant 0 \tag{3.2}$$

where $\Phi(\nu)$ is defined by

$$\Phi(\nu)=(\sqrt{9\nu^2+3\nu}-3\nu)\exp[\tfrac{1}{2}(9\nu+1-3\sqrt{9\nu^2+3\nu})] \tag{3.3}$$

then the Hadamard condition is satisfied for the Hart–Smith model (3.1) for arbitrary deformations.

In particular, inequalities (3.2) are satisfied for $d_2=0$ and $\nu=0.25$. In this case, from (1.5), (1.17), (2.4), and (2.6), we get

$$\begin{aligned}
\mu(\gamma)&=1+\gamma^2+\lambda^4(1-\gamma^2)^2(1-\gamma)^2\\
\Delta(\gamma)&=\lambda^4(1-\gamma)^2[\gamma^2+(\gamma^2-1)^2]\\
R(\gamma)&=-\lambda^4[\gamma^4(\gamma-1)(\gamma-2)+2\gamma(\gamma^2-1)+3\gamma^3+1]\\
\Sigma(\gamma)&=\lambda^4(1+\gamma)^2[\gamma^2+(1-\gamma\ )^4]
\end{aligned} \tag{3.4}$$

Since the inequalities $|(\gamma-1)(\gamma-2)| \geqslant \tfrac{1}{4}$, $\gamma^4 \leqslant 16$, and $3\gamma^3+1>4$ are satisfied for $\gamma \in (1, 2]$, condition (2.11) of Theorem 1 is satisfied. Requirements (2.12) and (2.13) are also satisfied (which can be easily verified using (3.4), taking into account that $\lambda=\gamma^{-1/2}$). Thus, for $d_2=0$ and $\nu=0.25$, a Hart–Smith material has high stiffness, i.e. a "limiting" thickness τ_* exists, above which a bifurcation of the equilibrium of a compressed bar is impossible. The value $\tau_*=0.7338$. was found by numerical methods.

We remark that expressions (3.4) imply that $\Delta(\gamma)>0$ $(\gamma>1)$. Thus, in view of (2.7)–(2.10), we get $\Gamma_\Delta^-=\phi$ and $\Gamma_G^+=\phi$, and, on the basis of assertion 5 of Theorem 1, we conclude that there are no symmetric buckling modes for any $\tau>0$. Besides, by assertion 2 of Theorem 1, T^- is equal to the interval $[0,\ \tau_*]$ in this case ($t_1^-=0$, since (2.16) is satisfied).

The results of computing the "limiting" thickness τ_* for a Hart–Smith material for various values of the elastic constants satisfying inequalities (3.2) are listed below ($n \equiv d_2/d_1$)

ν	3.0	5.0	6.0	8.0	10.0	20.0	20.0	20.0
n	0.2	0.3	0.99	0.99	0.99	0.99	0.4	0.1
τ_*	0.4298	0.4039	0.4042	0.3893	0.3788	0.3474	0.3408	0.3371

3.2. *A five-constant Alexander material* [5, 11]

$$\Pi = d_1 \int e^{k_1 (I_1 - 3)^2} dI_1 + d_2 (I_2 - 3) + d_3 \ln \frac{I_2 - 3 + k_2}{k_2}$$
$$(d_1 > 0,\ d_2 \geqslant 0,\ d_3 \geqslant 0,\ k_1 > 0,\ k_2 > 0) \tag{3.5}$$

The potential (3.5), which is in good agreement with the experimental results for chloroprene rubber [11], provides one more example of a material with "limiting" thickness. We remark that, apart from the notation, the energy (3.5) is identical with the Hart–Smith potential (3.1) for $d_2 = 0$ and $k_2 = 3$.

Lemma 2. Let the elastic constants in (3.5) satisfy at least one of the following sets of conditions (where Φ is defined by (3.3))

$$d_3 \leqslant 8 d_2 k_2 \tag{3.6}$$

$$\Phi(k_1) - \frac{3 d_3}{d_1 k_2^2} \geqslant 0, \quad k_2 \leqslant 6 \tag{3.7}$$

$$\Phi(k_1) - \frac{d_3}{4 d_1 (k_2 - 3)} \geqslant 0, \quad k_2 > 6 \tag{3.8}$$

Then the Alexander material (3.5) satisfies the Hadamard condition for any deformations.

The proof is omitted.

Calculating $\mu(\gamma)$ from (1.17) and (1.5), we obtain

$$\mu(\gamma) = 1 + \gamma^2 + 4 k_1 \lambda^4 (1 - \gamma^2)^2 \left[(\gamma - 1)^2 - \frac{n_2 \gamma^3}{2 k_1 r^2} e^{-s} \right] \left[1 + \left(n_1 + \frac{n_2 \gamma}{r} \right) e^{-s} \right]^{-1},$$
$$n_1 \equiv \frac{d_2}{d_1}, \quad n_2 \equiv \frac{d_3}{d_1}$$
$$r \equiv (\gamma - 1)^2 + k_2 \gamma, \quad s \equiv k_1 \lambda^4 (\gamma - 1)^4 \tag{3.9}$$

From (3.9) we can see that $\mu(\gamma) \sim 4 k_1 \gamma^4$ as $\gamma \to +\infty$, i.e. condition (2.13) of Theorem 1 is satisfied. The expressions for $R(\gamma)$, $\Delta(\gamma)$ and $\Sigma(\gamma)$ can be found with using (2.4), (2.6), and (3.9) (the expressions are awkward and will therefore be omitted). We will just mention that $\Sigma(\gamma) \to 4$ as $\gamma \to 1$ and requirement (2.12) is met. The constants d_1, d_2, d_3, k_1, k_2 can be chosen in such a way that condition (2.11) of Theorem 1 and at least one of the requirements (3.6)–(3.8) of Lemma 1 are satisfied. In particular, the relations

$$k_1 \geqslant 0.5, \quad d_2 \leqslant d_1, \quad d_3 = 0 \tag{3.10}$$

serve as sufficient conditions for $R(\gamma)$ to be negative for all $\gamma > 1$.

As a consequence, setting, for example, $d_1 = d_2 = k_1 = k_2 = 1$, $d_3 = 0$, one can simultaneously meet both the requirement (3.6) of Lemma 2 and the sufficient conditions (3.10) for $R(\gamma)$ to be negative.

Thus, for the specified values of the elastic constants, an Alexander material (3.5) has high stiffness.

We also remark that if restrictions (3.10) are satisfied, it can be verified directly that the function $\Delta(\gamma)$ satisfies the lower estimate

$$\Delta(\gamma) \geqslant (1 - \gamma)^2 [1 + \lambda^4 (\gamma^2 - 1)^2] \geqslant 0 \tag{3.11}$$

for $\gamma > 0$.

By Theorem 1, inequality (3.11) indicates that no symmetric buckling modes are possible in the case under consideration for any $\tau > 0$.

The results of computing the "limiting" thickness τ_* for several sets of elastic constants are listed below

k_1	5.0	10.0	15.0	20.0	20.0	20.0	20.0	40.0
k_2	28.0	6.0	5.0	3.0	3.0	3.0	9.0	20.0
n_1	30.0	20.0	5.0	1.0	2.0	3.0	40.0	1.0
n_2	10.0	3.9	2.0	0.99	0.99	0.99	3.0	100.0
τ_*	0.5514	0.4878	0.4191	0.3683	0.3815	0.3912	0.4665	0.3756

Note that in [12, 13] it was established on the basis of an approximate solution of the stability problem that a "limiting" thickness exists for Treloar and Mooney–Rivlin materials, which, in fact, is not the case. Indeed, the potentials corresponding to those materials have, respectively, the following form

$$\Pi = d(I_1 - 3), \qquad d = \text{const} > 0 \tag{3.12}$$

$$\Pi = d_1(I_1 - 3) + d_2(I_2 - 3), \qquad d_1, d_2 = \text{const} > 0 \tag{3.13}$$

In both cases we obtain

$$\mu(\gamma) = 1 + \gamma^2, \qquad R(\gamma) = \gamma^3 - 3\gamma^2 - \gamma - 1$$
$$\Delta(\gamma) = (\gamma - 1)^2, \qquad \Sigma(\gamma) = (\gamma + 1)^2 \tag{3.14}$$

on the basis of (1.5), (1.17), (2.4), and (2.6).

The Hadamard condition for models (3.12) and (3.13) holds for arbitrary deformations [1, 14]. Starting from the representation (3.14) for $R(\gamma)$ one can easily establish that $R(\gamma)$ has a unique zero $\gamma_0 = 3.38298$ on L with $R(\gamma) < 0$ for $\gamma \in (1, \gamma_0)$, and $R(\gamma) > 0$ for $\gamma > \gamma_0$. It follows that assumptions 1 and 2 of Theorem 3 are satisfied. The verification of assumptions 3–5 of this theorem using expressions (3.14) for $\Delta(\gamma)$ and $\Sigma(\gamma)$ is trivial. This means that all the requirements of Theorem 3 are met, i.e. the bar undergoes a loss of stability for all $\tau > 0$, contrary to the discussion in [12, 13] concerning the existence of a "limiting" thickness in these materials. By Theorem 3, symmetric buckling modes (with an arbitrary number $m \geq 1$ of nodal lines) exist for models (3.12) and (3.13) for any $\tau > 0$, but, since $\Delta(\gamma)$ is positive, they are always preceded by antisymmetric modes. This fact is consistent with the result obtained in [15].

3.3. *A hypothetical material with the potential*

$$\Pi = d_1 \int e^{k_1(I_1-3)^n} dI_1 + d_2 \int e^{k_2(I_2-3)^n} dI_2$$
$$(d_1 > 0, \ d_2 > 0, \ k_1 > 0, \ k_2 > 0, \ n > 0) \tag{3.15}$$

By using the method of [16], it can be shown that the Hadamard condition for model (3.15) is satisfied for arbitrary deformations. By (1.5) and (1.17), we find that

$$\mu(\gamma) = 1 + \gamma^2 + 2n(1+\gamma)^2 \frac{(d_1k_1 + d_2k_2)(\gamma - 1)^{2n}}{(d_1 + d_2)\gamma^n} \tag{3.16}$$

The expressions for $R(\gamma), \Delta(\gamma), \Sigma(\gamma)$ can be found using (2.4), (2.6), and (3.16). Without stating the expressions explicitly, we remark that $\Sigma(\gamma)$ satisfies (2.12), $\Delta(\gamma)$ is non-negative for all $\gamma > 0$, and the sufficient conditions for $R(\gamma)$ to be negative (for $\gamma > 0$) have the form

$$n \geq 1, \quad \frac{d_1k_1 + d_2k_2}{d_1 + d_2} > \frac{3^{n+1}}{n2^{2n+1}} \tag{3.17}$$

The first inequality in (3.17) is also necessary, since it can be shown that $R(\gamma)$ must have at least one zero in $(1, +\infty)$ for $n < 1$.

It follows from (3.16) that $\mu(\gamma)$ behaves like $M\gamma^{n+2}$ $(M = \text{const} > 0)$ as $\gamma \to +\infty$. Consequently, if the sufficient conditions (3.17) for $R(\gamma)$ to be negative are satisfied, all the conditions of Theorem 1 are satisfied for $n \geqslant 2$, and all the conditions of Theorem 2 are satisfied for $1 \leqslant n < 2$. If $n < 1$, then the case $\Gamma_R^0 \neq \phi$ occurs, i.e. the premise of Theorem 4 is true.

Thus, if $n \geqslant 2$, then the hypothetical material (3.15) has high stiffness, if $1 \leqslant n < 2$, it has moderate stiffness, and if $0 < n < 1$, it has low stiffness. Since $\Delta(\gamma) \geqslant 0$ $(\gamma > 0)$, no symmetric buckling modes are possible in the first two cases (in the third case, even though symmetric modes exist for all $\tau > 0$, they are preceded by antisymmetric modes).

We remark that the integral signs in (3.15) can, obviously, be omitted for $n = 1$

$$\Pi = d_1[e^{k_1(I_1-3)} - 1] + d_2[e^{k_2(I_2-3)} - 1] \tag{3.18}$$

For the potential (3.18), a sufficient condition for $R(\gamma)$ to be negative can be obtained from (3.17) by substituting $n = 1$

$$\frac{d_1k_1 + d_2k_2}{d_1 + d_2} > \frac{9}{8} \tag{3.19}$$

It follows that the energy (3.18) (subject to the restriction (3.19)) represents a material of moderate stiffness.

Example (3.15) demonstrates that the stiffness properties of a material (in the sense adopted in the present paper) are defined by the ability of the material to accumulate energy as the deformation increases. The greater the ability the more stable the material.

3.4. *A Klosner–Segal material* [5, 17]

$$\Pi = d_0(I_1 - 3) + d_1(I_2 - 3) + d_2(I_2 - 3)^2 + d_3(I_2 - 3)^3 \tag{3.20}$$

As has been shown experimentally [17], this model provides a satisfactory description of the behaviour of natural rubber for I_1, $I_2 < 8$. We shall confine ourselves to analysing the special case when $d_0 = 0$. In this case, the potential (3.20) can be shown to satisfy the Hadamard condition for arbitrary deformations if and only if the inequalities

$$d_1 \geqslant 0, \quad d_3 \geqslant 0, \quad 3d_2 + \sqrt{15 d_1 d_3} \geqslant 0 \tag{3.21}$$

are satisfied.

If the latter are supplemented by the condition $d_1 + d_3 \neq 0$, then Π will be automatically positive for $I_2 > 3$.

Using formulae (1.5) and (1.7), we find that

$$\mu(\gamma) = 1 + \gamma^2 + 4(1-\gamma^2)^2 \frac{d_2\gamma + 3d_3(\gamma-1)^2}{d_1\gamma^2 + 2d_2\gamma(\gamma-1)^2 + 3d_3(\gamma-1)^4} \tag{3.22}$$

The quantities $R(\gamma)$, $\Delta(\gamma)$ and $\Sigma(\gamma)$ can be expressed in terms of $\mu(\gamma)$ in accordance with (2.4), (2.6), and (3.22) (the explicit expressions are omitted due to lack of space). Under the restrictions adopted for the elastic constants, it can be shown that $R(\gamma)$ has at least one zero on $(1, +\infty)$. It follows that the conditions of Theorem 4 are satisfied and a Klosner–Segal material has low stiffness.

Numerical investigations reveal that in the special case when $d_1 = 60$, $d_2 = -10$, $d_3 = 1$ (this system of constants satisfies (3.21)) $R(\gamma)$ has an isolated zero at $\gamma_0 = 2.451325$ in $(1, +\infty)$. Moreover, inequalities (2.26) (in which one can take, for example, $\gamma_1 = 6$) are satisfied. The remaining assumptions of Theorem 3 are trivial to verify if one takes into account the equalities $\Delta(\gamma_0) \approx -4.64$, $\dot{R}(\gamma_0) \approx 12.36$, and $\lim_{\gamma \to 1+0} \Sigma(\gamma) = 4$ and uses Remark *3* on the theorem. The case under consideration is interesting in that relations (2.33)–(2.35) are satisfied for the critical values of γ. In particular, for $\tau = \tau_m^{(n)}$ (see formula (2.36)), γ_0 is a double eigenvalue of the boundary-value problem (1.4), (1.8), (1.9).

We observe that for $\gamma=\gamma_0$ we have $I_1=I_2=3.859267$, which remains within the boundaries of the experimentally established domain of applicability of potential (3.20).

We also remark that the whole discussion presented above, which applies to a Klosner–Segal material, remains valid for a Biderman material [5, 18] (the potential corresponding to the latter material can be obtained from (3.20) by making the substitution $I_1 \sim I_2$). The Biderman model provides an adequate description of the behaviour of rubber with sulphur filler [18].

4.5. *An essentially non-linear material with potential*

$$\Pi = d_1(I_1-3)^{\nu_1} + d_2(I_2-3)^{\nu_2} \tag{3.23}$$
$$(d_1 \geqslant 0,\ d_2 \geqslant 0,\ d_1^2+d_2^2 \neq 0,\ \nu_1 \geqslant 0.5,\ \nu_2 \geqslant 0.5)$$

For $\nu_1=\nu_2=1$ the energy (3.23) is identical with the Mooney–Rivlin potential (3.13). With the above-mentioned restrictions on the elastic constants, it can be shown that the material (3.23) satisfies the Hadamard condition for arbitrary deformations. A distinctive feature of the potential (3.23) is that, unless ν_1 and ν_2 are simultaneously equal to unity, the material is, in fact, non-linear even for very small deformations.

Here we shall confine ourselves to the special case when $\nu_1=\nu_2=\nu$. From (1.5), (1.17), (2.4), and (2.6) we find that

$$\begin{aligned}
&\mu(\gamma) = 1+\gamma^2+2(\nu-1)(1+\gamma)^2\\
&R(\gamma) = \gamma^3-(2\nu+1)\gamma^2-(4\nu-3)\gamma-(2\nu-1)\\
&\Delta(\gamma) = (1-\gamma)^2+2(\nu-1)(1+\gamma)^2, \quad \Sigma(\gamma) = (2\nu-1)(1+\gamma)^2
\end{aligned} \tag{3.24}$$

First we assume that $\nu = 0.5$. This being the case, a study of the function $R(\gamma)$ reveals that it has a unique zero γ_0 on the ray $(1, +\infty)$, and inequalities (2.26) (where $\gamma_1=+\infty$) as well as the condition $\dot{R}(\gamma_0)>0$ are satisfied. Furthermore

$$\operatorname{sign}\Delta(\gamma_0) = \operatorname{sign}\left(\nu-\frac{7}{8}\right) \tag{3.25}$$

Moreover, expression (3.24) for $\Sigma(\gamma)$ implies that (2.25) is satisfied. Hence it follows immediately that all the conditions of Theorem 3 are satisfied for $\nu>7/8$. If $\nu=7/8$, then (3.25) yields $\Delta(\gamma_0)=0$. This being so, it can be shown that γ_0 is equal to three and the relation $\lim_{\gamma\to 3} G(\gamma)=-2\pi/\sqrt{3}$ is satisfied. The latter relation indicates that $\gamma_0=3$ is not a limiting point of the set Γ_G^+. This means that the premise of Theorem 3 is also satisfied in the case in hand. Finally, for $\gamma<7/8$, we have $\Delta(\gamma_0)<0$ by (3.25). If (2.25) is taken into account and Remark 3 on Theorem 3 is used, then, as before, we can conclude that all the conditions of this theorem are met. Consequently, the material under consideration has low stiffness for $\nu>0.5$.

Now, let $\nu=0.5$. Then, by (3.24), we have $R(\gamma)=\gamma(\gamma-1)^2$, which implies that $\Gamma_R^+=L$, i.e. the premise of Theorem 4 is satisfied. Thus, for $\nu=0.5$ also, the potential (3.23) defines a material of low stiffness. The case $\nu=0.5$ is noteworthy in that some of the elementary inequalities, which are equivalent to the Hadamard condition [16], turn into equalities (for certain values of I_1, I_2). In other words, for $\gamma_1=\nu_2=0.5$, (3.23) belongs to the boundary of the space $H(\Pi)$ of potentials Π that satisfy the Hadamard condition. For small deformations of the non-distorted state, the behaviour of this material is similar to the deformation of a rigid-plastic body. We also remark that (2.43) is satisfied for $\nu=0.5$, i.e. for any thickness τ, the infimum of the set of critical values of γ is equal to unity. Thus the critical load q is non-zero.

In conclusion, we remark that the mathematical formalism developed in Sec. 2 can also be used to study the axisymmetric instability of a circular plate compressed by a uniform side pressure. All the qualitative properties revealed by the analysis of the behaviour of a bar remain valid for a plate. From the quantitative point of view, it is noteworthy that, according to a numerical experiment, the "limiting" thickness of a plate turns out to be two to three times smaller than that of a bar in the case of materials of high stiffness. As an example, we specify two values of τ_* computed for a plate made of a Hart–Smith material and two values for a plate made of an Alexander material, respectively

$$\nu = 3.0, \quad n = 0.2 \sim \tau_* = 0.1647;$$

$$\nu = 10.0, \quad n = 0.99 \sim \tau_* = 0.1458;$$

$$k_1 = 20.0, \quad k_2 = 3.0, \quad n_1 = 1.0, \quad n_2 = 0.99 \sim \tau_* = 0.1412;$$

$$k_1 = 20.0, \quad k_2 = 9.0, \quad n_1 = 40.0, \quad n_2 = 3.0 \sim \tau_* = 0.1747$$

It only remains to compare these values of τ_* with the data given in Examples 1 and 2.

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